Resit Exam
Mechanics & Relativity 2017–2018 (part 1b) and NAMR2
April 11 or 12, 2018

INSTRUCTIONS

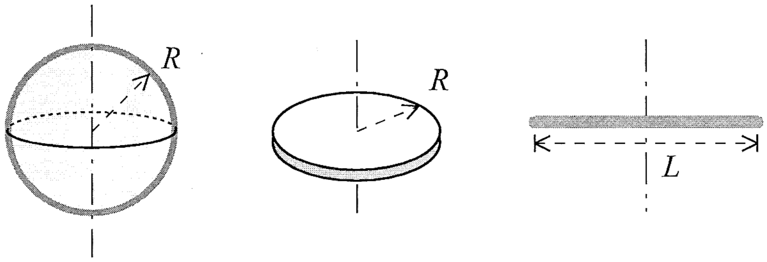
- Use a black or blue pen.
- This exam comprises 4 problems. Start your solution of each problem on a new page.
- The answers require clear arguments and derivations, all written in a well-readable manner. Scrap paper will not be graded and should not be submitted.
- The total number of points per problem is

Problem	# of points
1	6
2	5
3	8
4	9

and the grade is computed as (total # points) $/28 * 9 + 1$.

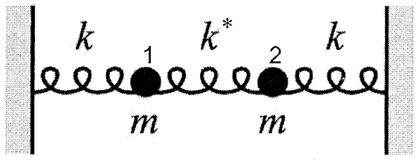
Some useful formulas

The mass of the thin-walled sphere, the thin disk and the rod shown below is M and distributed uniformly. The moments of inertia about the axis (shown by the dashed-dotted line) through their center of mass are:



$$I = \frac{2}{3}MR^2 \qquad I = \frac{1}{2}MR^2 \qquad I = \frac{1}{12}ML^2$$

Problem 1 (6 points) Consider two identical masses attached with springs with different spring constants denoted by k^* and k. Let us assume that the first mass is displaced by x_1 from its equilibrium and the second mass is displaced by x_2 from its equilibrium.



a. Show that the net force acting on mass 1 is given by

$$F = -kx_1 - k^* x_1 + k^* x_2 .$$

b. Show that the net force acting on mass 2 is given by

$$F = -kx_2 - k^* x_2 + k^* x_1 .$$

c. Show that the solution for $x_1 + x_2$ as a function of time can be expressed by

$$x_1 + x_2 = A_s \cos(\omega_s t + \phi_s),$$

where A_s is the amplitude, ϕ_s is the phase and ω_s is the frequency. Determine ω_s.

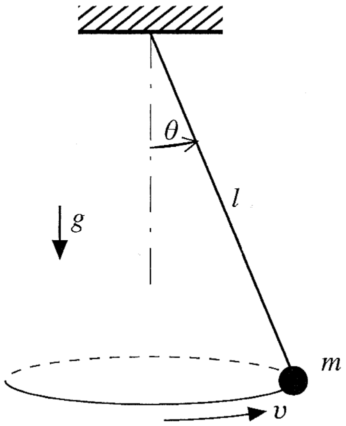
d. Similarly show that the solution for $x_1 - x_2$ can be expressed by

$$x_1 - x_2 = A_f \cos(\omega_f t + \phi_f),$$

where A_f is the amplitude, ϕ_f is the phase and ω_f is the frequency. Determine ω_f.

e. Find the motion of the two masses separately, i.e. $x_1(t)$ and $x_2(t)$.

Problem 2 (5 points) A mass m hangs from a massless string of length l under the influence of gravity. Conditions have been set up such that the mass goes around in a horizontal circle, with the string making a constant angle θ with the vertical. It is intuitively clear that the angle increases as the velocity v of the mass increases. The reason for this will be explored here by two observers: a stationary one and one who is co-rotating with the mass.



a. Draw the free-body diagrams of the mass (i.e. isolated from its environment, yet with all forces from the environment) as seen by the two observers.

b. Explain how both observers immediately conclude that the the tension in the string, T, is given by

$$T = mg/\cos\theta .$$

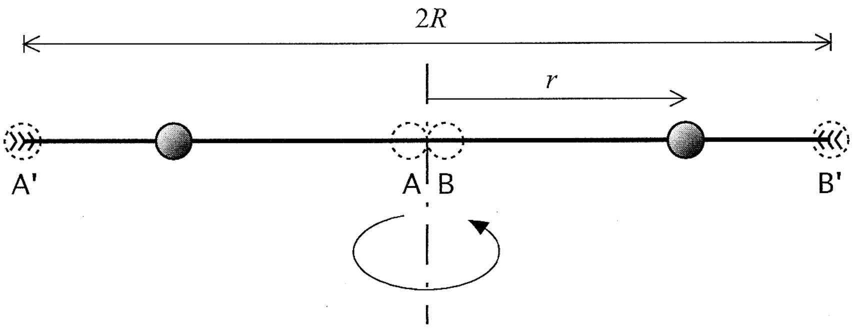
c. The co-rotating observer uses a similar argument to conclude that

$$T \sin\theta = mv^2/2\,,$$

with $r = l\sin\theta$. Explain.

d. Does the stationary observer agree with this observation? Explain the difference in the arguments.

Problem 3 (8 points)
Two balls of radius a and mass m can slide without friction along a horizontal rod A'B'. The rod has length $2R$ and mass M, and can rotate freely around a vertical axis through the center of the rod. Initially, the balls are kept close to the axis of rotation (positions AB) and the system spins with angular velocity ω_0. Since there is no friction, the rod+balls will rotate at the same speed, until, at a certain moment, the balls are released. Due to the centrifugal force they will both slide outwards and the angular velocity will change. When the balls hit the respective ends of the rod A'B', they get locked into position at the barbed ends* (indicated in the schematic as ⟫⟩ and ⟨⟪) as if it were a completely inelastic collision. In the pinned position the distance from the center of each ball to the axis is R.



a. Show that the moment of inertia of a solid sphere of mass m and radius a through its center is equal to $\frac{2}{5}ma^2$. Hint: make use of the moment of inertia of a thin-walled sphere, given under Useful Formulas on the front page.

b. Determine the moment of inertia of the system about the vertical axis of rotation when the centers of the balls are located at a distance r from this axis.

c. Compute the ratio between the angular velocity $\omega(R)$ at the moment that the balls have reached the ends of the rod and the initial value ω_0. Show that this ratio is equal to $1/7$ when $M = m$ and the size of the balls a can be neglected relative to R.

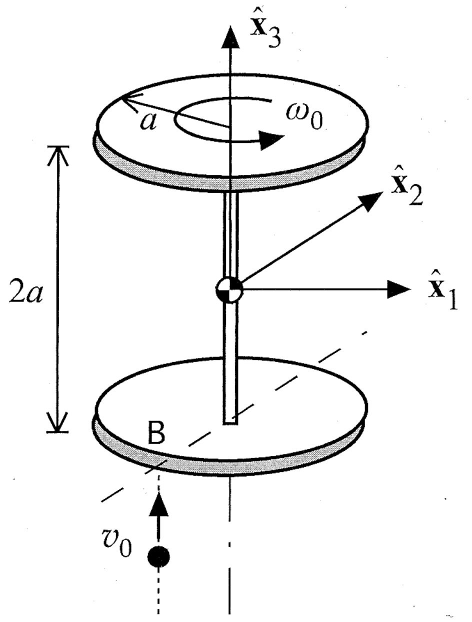
d. How much energy gets dissipated when the balls are pinned at the ends?

*NL: weerhaken

Problem 4 (9 points)

From a dynamical point of view, a satellite of mass m can be regarded as a set of two identical disks of radius a. The disks are connected by a massless rod of length $2a$. The satellite spins freely in space around its axis with angular velocity ω_0.

At a certain moment, a meteorite with velocity v_0 impacts and gets attached to the satellite at the edge of one of the disks (point B). The mass of the meteorite, m_0, is so much smaller than that of the satellite, that it does not influence the moment of inertia, yet the *in*-elastic collision induces a precession motion. In order to describe this motion, we introduce a set of base vectors at the center of mass of the satellite, with $\hat{\mathbf{x}}_3$ along the axis of initial rotation, $\hat{\mathbf{x}}_2$ parallel to the disks and pointing opposite to the location of impact, and $\hat{\mathbf{x}}_1$ completing a right-handed system.



a. Argue why $\hat{\mathbf{x}}_i$ $(i = 1, 2.3)$ are principal axes and show that the principal moments of inertia are given by (hint: make use of the moments of inertia of a disk)

$$I_1 = I_2 = \frac{5}{4}ma^2\,, \quad I_3 = \frac{1}{2}ma^2\,.$$

In the sequel, use the shorthand $I \equiv I_1 = I_2$.

b. The spin vector prior to impact is $\boldsymbol{\omega} = \omega\hat{\mathbf{x}}_3$. Compute the spin vector after the blow.

c. Determine the orientation of the precession axis relative to the satellite in a clear graph.

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Problem 1

a. Force on mass 1 by moving x_1 while keeping x_2 fixed is : $F_a = -kx_1 - k^*x_1$ Now Force acting on mass 1 by moving x_2 while holding x_1 fixed is : $F_b = k^*x_2$. The net force acting on mass 1 is then $F = F_a + F_b = -(k+k^*)x_1 + k^*x_2$. [1]

b. Similarly we can work out for mass 2 also, and the net force will be $F = -(k + k^*)x_2 + k^*x_1$. [1]

c. Since,

$$m\ddot{x}_1 = -(k+k^*)x_1 + k^*x_2$$
$$m\ddot{x}_2 = -(k+k^*)x_2 + k^*x_1$$

therefore,

$$m(\ddot{x}_1 + \ddot{x}_2) = -k(x_1 + x_2)$$
$$x_1 + x_2 = A_s\cos(\omega_s t + \phi_s), \qquad \omega_s = \sqrt{\frac{k}{m}}$$

[1]

d. Similarly, we can show that

$$m(\ddot{x}_1 - \ddot{x}_2) = (-k - 2k^*)(x_1 - x_2)$$
$$x_1 - x_2 = A_f\cos(\omega_f t + \phi_f), \qquad \omega_f = \sqrt{\frac{k+2k^*}{m}}$$

[1]

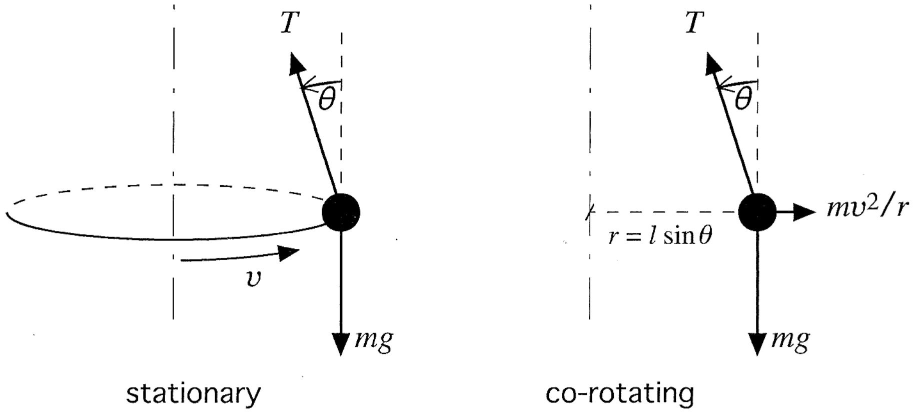
e. The normal modes are then x_1 and x_2

$$x_1 = \frac{1}{2}[(x_1 + x_2) + (x_1 - x_2)] = \frac{1}{2}[A_s\cos(\omega_s t + \phi_s) + A_f\cos(\omega_f t + \phi_f)]$$
$$x_2 = \frac{1}{2}[(x_1 + x_2) - (x_1 - x_2)] = \frac{1}{2}[A_s\cos(\omega_s t + \phi_s) - A_f\cos(\omega_f t + \phi_f)]$$

[2]

Problem 2

a.



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b. Because of vertical equilibrium: $mg - T\cos\theta = 0$. 1

c. Horizontal equilibrium of forces, because the mass is at rest for the co-rotating observer: $mv^2/r - T\sin\theta = 0$. 1

d. Yes, since from the vantage point of the stationary observer, the mass needs to have a centripetal acceleration $a_{\text{cent}} = v^2/r$ in order for it to circle around. This acceleration is supplied by the force pointing inwards: $T\sin\theta = ma_{\text{cent}}$. 1

Problem 3

a. The moment of inertia of a solid sphere is the sum of infinitesimal moments of inertia dI of a spherical shell of mass dm and radius $r \in [0, a]$: $dI = \frac{2}{3}r^2 dm$. The mass dm is determined by the mass density $\rho = m/(\frac{4}{3}\pi a^3)$ and the volume of the shell, $4\pi r^2 dr$. Putting this information together, we find

$$I = \int_0^r \frac{2}{3} r^2 \frac{m}{\frac{4}{3}\pi a^3} 4\pi r^2 dr = 2\frac{m}{a^3}\int_0^r r^4 dr = \frac{2}{5}ma^2.$$

qed 1

b. The moments of inertia of the components are given by:

$$I_{\text{rod}} = \frac{1}{12}M(2R)^2 = \frac{1}{3}MR^2\,, \quad I_{\text{ball}} = \frac{2}{5}ma^2$$

Making use of the parallel axis theorem for the balls at position r, we find in total

$$I(r) = \frac{1}{3}MR^2 + 2m\left(\frac{2}{5}a^2 + r^2\right).$$

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c. Conservation of angular momentum requires:

$$I(r)\omega(r) = \text{constant}$$

implying that

$$I(a)\omega_0 = I(R)\omega(R) \Longrightarrow \frac{\omega(R)}{\omega_0} = \frac{I(a)}{I(R)}\,.$$

When $M = m$ and $a << R$ this gives

$$I(a) \approx \frac{1}{3}MR^2\,, \quad I(R) = \frac{1}{3}MR^2 + 2MR^2 = \frac{7}{3}MR^2$$

and

$$\frac{\omega(R)}{\omega_0} = \frac{1}{7}$$

as requested. [3]

d. The amount of dissipated energy ΔE is the difference between the initial energy (when the balls do not yet have a radial velocity) and the energy after the balls have been clamped (and lost their radial kinetic energy):

$$\begin{aligned}\Delta E &= \frac{1}{2}I(a)\omega_0{}^2 - \frac{1}{2}I(R)\omega^2(R)\\ &= \frac{1}{2}I(a)\omega_0{}^2 - \frac{1}{2}\omega_0{}^2 I^2(a)/I(R) = \frac{1}{2}I(a)\omega_0{}^2[1 - I(a)/I(R)]\,.\end{aligned}$$

In the limiting case $M = m$ and $a << R$: $\Delta E = \frac{1}{7}MR^2$. [2]

Problem 4

a. The $\hat{\mathbf{x}}_i$ $(i = 1, 2.3)$ denote principal axes because $\hat{\mathbf{x}}_3$ coincides with an axis of axial symmetry, and because the satellite is symmetric about both the $\hat{\mathbf{x}}_1$ and the $\hat{\mathbf{x}}_2$ axes.

The moment of inertia of a disk with radius r and mass m about its axis is $I_z = \frac{1}{2}mr^2$. As I_x en I_y (in the CM of the disk) are equal and satisfy $I_x + I_y = I_z$, we have $I_x = I_y = mr^2/4$.

The moment of inertia of the satellite about its axis of rotation therefore is $I_3 = 2\frac{1}{2}(m/2)a^2 = \frac{1}{2}ma^2$. The other two principal moments of inertia are equal to the I_x of two disks, each shifted by a distance a away from the CM:

$$I_1 = I_2 = 2\left[\frac{1}{4}(m/2)a^2 + (m/2)a^2\right] = \frac{5}{4}ma^2\,.$$

qed [3]

b. This problem needs to be analysed by conservation of angular momentum (energy is not conserved during the inelastic impact). Angular momentum (with respect to the CM) prior to impact is:

$$\mathbf{L}_{\text{before}} = \mathbf{I}\boldsymbol{\omega} + \mathbf{L}_{\text{meteorite}} \,,$$

where the first term is simply equal to $\mathbf{I}\boldsymbol{\omega} = I_3\omega\hat{\mathbf{x}}_3$ because $\hat{\mathbf{x}}_i$ $(i = 1, 2.3)$ are principal axes. The contribution of the meteorite before impact can formally be computed from $\mathbf{L}_{\text{meteorite}} = \mathbf{r} \times \mathbf{p}$ with $\mathbf{r} = -a\hat{\mathbf{x}}_2$ and $\mathbf{p} = m_0 v_0 \hat{\mathbf{x}}_3$, or by using the geometric insight that its angular momentum points in the $-\hat{\mathbf{x}}_1$ direction (in accordance with the right-hand rule) and has magnitude $m_0 v_0 a$: 1

$$\mathbf{L}_{\text{voor}} = I_3\omega\hat{\mathbf{x}}_3 - m_0 v_0 a \hat{\mathbf{x}}_1 \,. \tag{1}$$

To compute the angular momentum after impact, we use the information that the meteorite has not changed the moment of inertia; the spin vector, however, is still unknown. Denoting the spin component by $\{\omega_1, \omega_2, \omega_3\}$, angular momentum after impact is given by 1

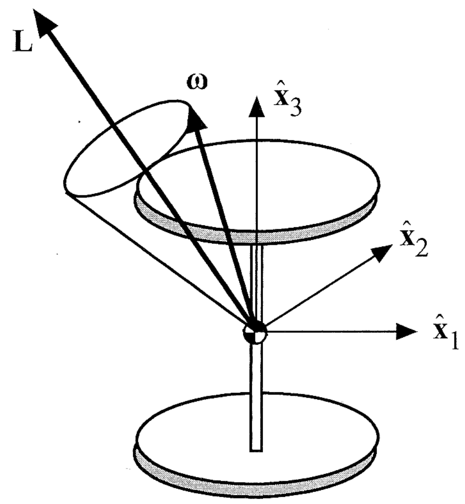
$$\mathbf{L}_{\text{after}} = I\omega_1\hat{\mathbf{x}}_1 + I\omega_2\hat{\mathbf{x}}_2 + I_3\omega_3\hat{\mathbf{x}}_3 \,.$$

The condition $\mathbf{L}_{\text{after}} = \mathbf{L}_{\text{before}}$ allows finding each of the spin components individually, yielding 0.5

$$\begin{aligned}\boldsymbol{\omega} &= -\frac{m_0 v_0 a}{I}\hat{\mathbf{x}}_1 + \omega\hat{\mathbf{x}}_3 \\ &= -\frac{4}{5}\frac{m_0 v_0}{ma}\hat{\mathbf{x}}_1 + \omega\hat{\mathbf{x}}_3 \,.\end{aligned}$$

The physical interpretation is that the spin about its original axis of rotation does not change because the satellite is much heavier than the meteorite. Yet, the meteorite does induce a spin perpendicular to the it; the magnitude of this spin is not necessarily small: it depends on v_0! 1.5

c. Angular momentum is conserved, so $\mathbf{L}$ is a fixed vector in space. From a fixed point in space it therefore looks as if the satellite rotates around $\mathbf{L}$ (with $\boldsymbol{\omega}$ in between $\mathbf{L}$ and $\hat{\mathbf{x}}_3$ because $I_3 < I$). During this motion, the spin vector traces a cone in space as indicated below. The orientation of the precession axis with respect to the satelite is controlled by the angle between $\mathbf{L}$ and $\hat{\mathbf{x}}_3$.



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